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## Renormalization group in quantum mechanics

Pierre Gosselin<sup>†</sup> and Hervé Mohrbach<sup>‡</sup>

<sup>†</sup> Université Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, UFR de Mathématiques, BP74, 38402 Saint Martin d'Hères Cedex, France

<sup>‡</sup> LPLI Institut de Physique, 1 boulevard D Arago, F-57070 Metz, France

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**Abstract.** We establish the renormalization group (RG) equation for the running action in the context of a one-quantum-particle system. This equation is deduced by integrating one Fourier mode after another in the path integral formalism. It is free from the well known pathologies which appear in quantum field theory due to the sharp cutoff. We show that for an arbitrary background path the usual local form of the action is not preserved by the flow. To cure this problem we consider a more general action than usual, which is stable in the RG flow. This allows us to obtain a new consistent RG equation for the action.

### 1. Introduction

The renormalization group (RG) [1] is a powerful tool for investigating problems with many degrees of freedom as in quantum field theory or in the statistical mechanics of phase transitions. Its advantage is in being non-perturbative by nature. The RG equation is a flow equation for an effective action  $S_\Lambda[\varphi(x)]$  on lowering the ultraviolet cutoff  $\Lambda$ :

$$\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda[\varphi] = \mathcal{F}[S_\Lambda].$$

The Wegner–Houghton equation [2] is a kind of RG flow equation in momentum space where the cutoff  $\Lambda$  is the highest momentum of the field. This equation has been widely studied in the past for problems related to phase transitions (fixed points, computation of the critical exponents etc).

The RG is not limited to statistical mechanics or field theory but is also useful for extracting non-perturbative results in quantum mechanics. In [3], we derived the RG equation in momentum space for the (arbitrary) potential part of a one-quantum-particle action at zero and finite temperature (at zero temperature it corresponds to the one-dimensional version of the Wegner–Houghton equation). This equation allowed us to compute, as an example, the ground state and first excited energy level of the anharmonic oscillator with great accuracy. The RG equation was deduced by integrating one mode after the other in the path integral. Starting with an initial potential  $V(x(t))$ , we obtain at each step of the integration an effective potential  $V_m(x(t))$ . Actually the computation was performed by Taylor expanding the potential  $V_m(x(t))$  containing a path  $x(t)$  with  $m$  modes around the zero-mode path  $x_0$  (called in this paper the constant-background path in analogy with the constant-background field in field theory) and by integrating only the last mode  $x_m$  in the path integral. In this manner we obtain an equation relating  $V_m(x_0)$  and  $V_{m-1}(x_0)$ . The approximation made by projecting on the zero-component path  $x_0$  is named the local potential approximation (LPA) [4]. In contrast to the statistical case,

where we are looking for the fixed point of the RG equation, we were interested in integrating this equation until the zero mode ( $m = 0$ ), in order to obtain the so-called effective potential  $V_0(x_0)$ . At zero temperature the effective potential is equal to the ground state energy [5], so our procedure was applied to the computation of the ground state energy of the anharmonic oscillator. This procedure is summarized in section 2.

It is the aim of this paper to extend the previous RG analysis beyond the LPA by considering a quantum particle in an arbitrary potential for an arbitrary background path. This allows us to compute the RG equation not only for the potential but for the complete action. It is shown that a local potential is no longer consistent because locality is not preserved by the RG flow. We then choose a particular non-local action, which is shown to be preserved along the RG flow but only in the zero-temperature limit. The finite-temperature case is then not considered in this paper. In addition it is shown that the kinetic term with a path-independent mass term is kept constant during the flow (in contrast to other papers in field theory), whereas a path-dependent mass term obeys an RG equation too.

These results are in contradiction with other works in field theory [6–8], where the authors found flow equations for the potential and the kinetic energy for arbitrary background field, preserving the locality assumption. In fact, only the equation for a constant-background field yielding the flow for the potential is well established. It is well known that the natural extension of this approximation to non-constant background fields obtained by considering a derivative expansion of the action suffers pathologies (non-analyticity) due to the sharp cutoff used in the computation [4, 6, 9]. In particular the anomalous dimension cannot be computed with certitude. Therefore the results obtained in field theory with a sharp cutoff must be taken with care. On the other hand the introduction of a smooth cutoff formalism [4, 9] gives a well defined mathematical cadre for the computation but the physical quantities depend on the parameters characterizing the smoothness, the so-called scheme dependence [10].

In this paper our computation in quantum mechanics (one-dimensional field theory) is free of the just discussed pathologies if we take the limit of infinite volume (in our case zero temperature) at the end of the computation, whereas they are present if we start directly in the continuum. This shows that the generation of non-analytical terms is in fact only an artifact of the computation in the continuum and disappears if we perform a rigorous discrete calculus.

This paper is organized as follows: we present the general formalism in the context of the LPA approximation in section 2. In section 3, by deriving the flow of the complete action we find that a new kind of potential must be necessarily introduced to obtain a closed form for the action along the flow. In section 4, we deduce in addition some new equations for the flow of the coupling constants. The analysis of these flows proves that the LPA is enough to compute the ground state energy exactly (in the case of a constant mass term). The RG equation for the kinetic energy is also computed in section 5, and discussed in view of field theory applications in section 6.

## 2. RG equations of the potential for a constant-background path

We work in the Euclidean formalism at a finite temperature and discrete time and quickly retrieve the RG equation for a constant-background path as was done in [3], but here we limit ourself to the zero-temperature limit, so we will always neglect contributions of order  $\frac{1}{\beta^2}$ . Keeping a finite temperature in the intermediary steps allows us to work with a finite number of Fourier modes, so that we can integrate one mode after another in the path integral.

Consider the Euclidean action of a quantum particle at a finite temperature

$$S(x) = \int_0^{\hbar\beta} \left( \frac{1}{2} M \left( \frac{d}{dt} x(t) \right)^2 + V(x(t)) \right) \tag{1}$$

with  $M$  the mass,  $V$  the potential.

The effective potential is defined as a constrained path integral over periodic paths with period  $\hbar\beta$

$$\exp(-\beta V_0(x_0)) = \int \mathcal{D}x \delta(\bar{x} - x_0) \exp\left(-\frac{1}{\hbar} S(x)\right) \tag{2}$$

where  $\bar{x} = \frac{1}{\hbar\beta} \int_0^{\hbar\beta} dt x(t)$  is the average position of the particle in the time interval  $t \in [0, \beta]$ .

We consider the Feynman path integral with a discretized time  $t_n = \frac{nT}{N+1} = n\epsilon$ , with  $N$  an arbitrary large number, and  $n = 0, \dots, N + 1$ . The Fourier decomposition of a periodic path  $x(t_n)$  contains only a finite number of Fourier modes

$$x(t_n) = x_0 + \frac{1}{\sqrt{N+1}} \sum' \exp(i\omega_m t_n) x_m + \text{h.c.} \tag{3}$$

where  $\sum'$  is from 1 to  $\frac{N}{2}$  if  $N$  is even and from 1 to  $\frac{N-1}{2}$  if  $N$  is odd. The  $x_m$  are the Fourier modes and  $\omega_m^2 = \frac{2-\cos\frac{2\pi m}{N+1}}{\epsilon^2}$ . The discrete action is

$$S_{\frac{N}{2}}(x) = \epsilon \sum_0^{\frac{N}{2}} M \omega_m^2 |x_m|^2 + \epsilon \sum_{n=1}^{N+1} V_{\frac{N}{2}}(x(t_n)). \tag{4}$$

The partition function is

$$Z = \int \frac{dx_0}{\sqrt{\frac{2\pi\hbar\epsilon}{M}}} \int \prod_1^{\frac{N}{2}} \frac{dx_m d\bar{x}_m}{\frac{2\pi\epsilon\hbar}{M}} \exp\left(-\frac{1}{\hbar} S_{\frac{N}{2}}\right). \tag{5}$$

Now using the fact that  $\prod_1^{\frac{N}{2}} \epsilon^2 \omega_m^2 = \sqrt{N+1}$  (see [5]) and  $\hbar\beta = (N+1)\epsilon$ , we can drop the first integral to obtain the effective potential

$$\exp(-\beta V_0(x_0)) = \int \prod_1^{\frac{N}{2}} \frac{dx_m d\bar{x}_m}{\frac{2\pi\epsilon\hbar}{\epsilon^2 \omega_m^2 M}} \exp\left(-\frac{1}{\hbar} S_{\frac{N}{2}}\right). \tag{6}$$

The effective potential is then obtained after integration on all the Fourier modes except the zero mode, so that it accounts for all the quantum fluctuations. At zero temperature it is clear that the partition function is dominated by the minimum of  $V_0(x_0)$ , which lies at  $x_0 = 0$  in the absence of a phase transition. Then the ground state energy is given by  $E_0 = V_0(0)$ . Instead of performing the integration on all the Fourier modes in one step to compute the effective potential, which is in general a very difficult task and needs some approximations, we integrate only one mode to obtain a new action with one mode less. This is the spirit of the RG. Starting with  $V_{\frac{N}{2}}(x(t))$  as the initial potential we denote by  $V_m$  the running potential obtained after  $N - m$  integrations. To find the potential  $V_{m-1}$  with respect to  $V_m$  we consider paths with only one Fourier mode:

$$x(t_n) = x_0 + \frac{1}{\sqrt{N+1}} \exp(i\omega_m t_n) x_m + \text{h.c.} \tag{7}$$

By integrating on the mode  $x_m$  we will obtain a new potential  $V_{m-1}$  which could depend only on the zero mode  $x_0$  due to the particular path chosen. We could also choose a path different

from (7). Then the potential  $V_{m-1}$  would be a function of another path. The particular path chosen is called the background path, and for this reason  $x_0$  is called the constant-background path.

Then by definition we have the relation

$$\begin{aligned} \exp(-\beta V_{m-1}(x_0)) &= \int \frac{dx_m d\bar{x}_m}{\frac{2\pi\epsilon\hbar}{\epsilon^2\omega_m^2 M}} \\ &\times \exp\left(-\frac{\epsilon}{\hbar}\left(M\omega_m^2|x_m|^2 + \sum_{n=0}^{N+1} V_m\left(x_0 + \frac{\exp(i\omega_m t_n)x_m}{\sqrt{N+1}} + \text{h.c.}\right)\right)\right). \end{aligned} \quad (8)$$

Expanding the potential  $V_m$  to second order around the point  $x_0$ , and summing over  $n$ , yields

$$\begin{aligned} \exp(-\beta V_{m-1}(x_0)) &= \exp(-\beta V_m(x_0)) \int \frac{dx_m d\bar{x}_m}{\frac{2\pi\epsilon\hbar}{\epsilon^2\omega_m^2 M}} \\ &\times \exp\left(-\frac{\epsilon}{\hbar}(M\omega_m^2 + V_m^{(2)}(x_0))|x_m|^2 + \mathcal{O}\left(\frac{1}{\beta^2}\right)\right). \end{aligned}$$

After the Gaussian integration on the modes  $x_m$  and  $\bar{x}_m$  we find the following RG equation for the potential:

$$V_{m-1}(x_0) = V_m(x_0) + \frac{1}{\beta} \log\left(1 + \frac{V_m^{(2)}(x_0)}{M\omega_m^2}\right) + \mathcal{O}\left(\frac{1}{\beta^2}\right) \quad (9)$$

where  $V_m^{(2)}(x_0) = \frac{d^2}{dx_0^2} V_m(x_0)$ .

This equation is the one-dimensional version of the Wegner–Houghton equation in the LPA. It is this equation which has been studied in the past in the statistical mechanics of phase transitions. Going beyond this approximation is notoriously difficult as discussed in the introduction. We will return to this question in the last section. Only in the  $D = 1$  case is the computation free from of trouble as shown in the following.

As discussed in [3], it is very difficult to integrate numerically the nonlinear equation (9) in order to find the ground state energy. It is much easier to perform a Taylor expansion of the running potential in order to compute the flow of the coupling constants. Unfortunately we have to truncate this expansion and the results cannot be exact, but are nevertheless impressively good (see [3]). Defining the  $n$ th coupling constant at the scale  $m$  by  $g_m^{(n)}(x_0) = \frac{d^n}{dx_0^n} V_m(x_0)$  we can obtain the flow of all the coupling constants by deriving (9). Then the Taylor expansion of the potential  $V_m(x(t_n))$  around  $x_0$  yields the following expansion of the interaction in the Fourier space at the cutoff scale  $m$ :

$$\begin{aligned} \frac{\epsilon}{\beta} \sum_{n=1}^{N+1} V_m(x(t_n)) &= g_m^{(0)}(x_0) + \frac{g_m^{(2)}(x_0)}{2!(N+1)} \sum_{\substack{p=-m \\ p \neq 0}}^m x_p x_{-p} \\ &+ \frac{g_m^{(4)}(x_0)}{4!(N+1)^2} \sum_{p_1, \dots, p_4}^m \delta_{p_1 + \dots + p_4, 0} x_{p_1} x_{p_2} x_{p_3} x_{p_4} + \dots \end{aligned} \quad (10)$$

where  $g_m^{(0)}(x_0) = V_m(x_0)$ . It will be shown below that this expansion is actually not preserved by the RG for a non-constant-background path. A more general potential still local in time must be introduced to avoid inconsistencies which appear when one tries to deduce the RG equation for an non-constant-background path.

### 3. RG equations of the potential for an arbitrary background path

In this section we keep the background path arbitrary. Instead of computing  $V_{m-1}(x_0)$  we want to deduce the running action  $S_{m-1}(x^{(m-1)})$  from the action  $S_m(x^{(m)})$ , where we define by

$$x^{(m)}(t_n) = x_0 + \frac{1}{\sqrt{N+1}} \sum_{p=1}^m \exp(i\omega_p t_n) x_p + \text{h.c.} \quad (11)$$

the truncated path with Fourier components up to  $m$ .

Analogously to the preceding section one defines an action at step  $m-1$  by integrating only on the two modes  $x_m$  and  $\bar{x}_m$ ; that is,

$$\exp\left(-\frac{1}{\hbar} S_{m-1}(x^{(m-1)})\right) = \int \frac{dx_m d\bar{x}_m}{\epsilon^2 \omega_m^2 M} \exp\left(-\frac{1}{\hbar} S_m(x^{(m)})\right). \quad (12)$$

To obtain only the contributions of order  $\frac{1}{\beta}$  it is enough to expand the action  $S_m(x^{(m)})$  around  $x^{(m-1)}$  to the second order [3], so that the result is obtained after a Gaussian integration:

$$S_{m-1}(x^{(m-1)}) = S_m(x^{(m-1)}) + \frac{1}{2} \log(\det(\mathcal{A})) - J^t \mathcal{A}^{-1} J \quad (13)$$

with  $\mathcal{A}$  the symmetric  $2 \times 2$  matrix

$$\mathcal{A} = \begin{pmatrix} 1 + \sum_{n=0}^{N+1} \frac{V_m^{(2)}(x^{(m-1)}(t_n))(1+\cos(2\omega_m t_n))}{(N+1)\omega_m^2 M} & - \sum_{n=0}^{N+1} \frac{V_m^{(2)}(x^{(m-1)}(t_n)) \sin(2\omega_m t_n)}{(N+1)\omega_m^2 M} \\ - \sum_{n=0}^{N+1} \frac{V_m^{(2)}(x^{(m-1)}(t_n)) \sin(2\omega_m t_n)}{(N+1)\omega_m^2 M} & 1 + \sum_{n=0}^{N+1} \frac{V_m^{(2)}(x^{(m-1)}(t_n))(1-\cos(2\omega_m t_n))}{(N+1)\omega_m^2 M} \end{pmatrix} \quad (14)$$

and

$$J = \begin{pmatrix} \frac{\epsilon}{\sqrt{N+1}} \sum_{n=0}^{N+1} V_m^{(1)}(x^{(m-1)}(t_n)) \cos(\omega_m t_n) \\ - \frac{\epsilon}{\sqrt{N+1}} \sum_{n=0}^{N+1} V_m^{(1)}(x^{(m-1)}(t_n)) \sin(\omega_m t_n) \end{pmatrix}. \quad (15)$$

Now we consider a mass term which is position independent. As we shall see *a posteriori*, this term will stay constant during the RG flow (*a priori* there was a possibility of obtaining a flowing mass  $M_m$  as well as generation of higher-derivative interactions). Then we write only the equation for the running potential

$$\begin{aligned} \epsilon \sum_{n=0}^{N+1} V_{m-1}(x^{(m-1)}(t_n)) &= \epsilon \sum_{n=0}^{N+1} V_m(x^{(m-1)}(t_n)) + \frac{1}{2} \log \left( \left( 1 + \sum_{n=0}^{N+1} \frac{V_m^{(2)}(x^{(m-1)}(t_n))}{(N+1)\omega_m^2 M} \right)^2 \right. \\ &\quad \left. - \sum_{n_1=0}^{N+1} \sum_{n_2=0}^{N+1} \frac{V_m^{(2)}(x^{(m-1)}(t_{n_1}))}{(N+1)\omega_m^2 M} \frac{V_m^{(2)}(x^{(m-1)}(t_{n_2}))}{(N+1)\omega_m^2 M} \cos(2\omega_m(t_{n_2} - t_{n_1})) \right) \\ &\quad - J^t \mathcal{A}^{-1} J. \end{aligned} \quad (16)$$

Expanding  $V_m^{(2)}$  around  $x_0$  in the left-hand side of (16) does not generate a flow for the kinetic term, so the logarithm term contributes only to the potential (we can check that this is also true for the ‘source’ term). The explicit form of  $J^t \mathcal{A}^{-1} J$  is not necessary for the point we want to show. The problem is the appearance of the non-local double sum in the logarithm

$$\sum_{n_1=0}^{N+1} \sum_{n_2=0}^{N+1} \frac{V_m^{(2)}(x^{(m-1)}(t_{n_1}))}{(N+1)\omega_m^2 M} \frac{V_m^{(2)}(x^{(m-1)}(t_{n_2}))}{(N+1)\omega_m^2 M} \cos(2\omega_m(t_{n_2} - t_{n_1})) \quad (17)$$

which makes it impossible to obtain a local potential for  $V_{m-1}$ . That is, if we have a local potential of the form

$$V_m(x^{(m)}) = V_m \left( x_0 + \frac{1}{\sqrt{N+1}} \sum_{p=1}^m \exp(i\omega_p t_n) x_p + \text{h.c.} \right) \quad (18)$$

the potential at the scale  $m - 1$  cannot be written:

$$V_{m-1}(x^{(m-1)}) = V_{m-1}\left(x_0 + \frac{1}{\sqrt{N+1}} \sum_{p=1}^{m-1} \exp(i\omega_p t_n) x_p + \text{h.c.}\right). \tag{19}$$

In other words the assumption concerning the expansion of the running potential (10) is too strong and this kind of expansion is not preserved by the RG. Another way to see this problem is to replace  $V_m$  in the right-hand side of (16) by its expansion (10). It is then easy to check that the expansion obtained for  $V_{m-1}$  is of a different kind from (10). Actually more coupling constants are introduced. We deduce that the usual expansion (10) is not preserved along the RG flow.

A consequence of this unclosedness can easily be seen by developing the RG equation (16) around  $x_0$  to compute the equation for the second derivative of the potential at the point  $x_0$ . The zero order of this expansion gives the RG equation for the running potential

$$V_{m-1}(x_0) = V_m(x_0) + \frac{1}{\beta} \log\left(1 + \frac{V_m^{(2)}(x_0)}{\omega_m^2 M}\right)$$

which is the usual one (9), whereas the second order gives

$$V_{m-1}^{(2)}(x_0) = V_m^{(2)}(x_0) + \frac{1}{\beta} \frac{V_m^{(4)}(x_0)}{\omega_m^2 M + V_m^{(2)}(x_0)}. \tag{20}$$

These two equations are evidently inconsistent with each other. This is again a manifestation of the fact that the local ansatz for the potential at the scale  $m$  is not preserved after one step of the RG, and a kind of non-locality has to be introduced.

#### 4. Generalized potential

To avoid the just-mentioned inconsistency, we have to consider a class of potentials which is preserved by the RG flow. Looking to the RG equation leads us to consider for the potential  $V_m$  a function of the  $m + 1$  independent variables  $V_m(x_0, \dots, x_p e^{i\omega_p t} + x_{-p} e^{i\omega_p t}, \dots, e^{i\omega_m t} x_m + e^{-i\omega_m t} x_{-m})$ , denoted again for convenience by  $V_m(x^{(m)}(t))$ .

In fact, due to the nonlinearity of the  $\omega_m$ , this kind of potential is not preserved by the RG flow, but recall that we want to consider the limit  $\beta \rightarrow \infty$ . In such a limit  $\omega_m = \frac{2\pi m}{n\beta}$ , and we will show that the class of function considered is preserved by the flow.

To derive the new flow equation for the potential, it is easier to work in functional space. We write the action at scale  $m$

$$S_m(x^{(m)}) = \epsilon \sum_{p=0}^m M \omega_p^2 |x_p|^2 + \beta U_m(x^{(m)}) \tag{21}$$

where we have introduced the notation

$$U_m(x^{(m)}) = \frac{\epsilon}{\beta} \sum_{n=0}^{N+1} V_m(x_0, e^{i\omega_1 t} x_1 + e^{-i\omega_1 t} x_{-1}, \dots, e^{i\omega_m t} x_m + e^{-i\omega_m t} x_{-m}). \tag{22}$$

Expanding the potential around  $x_0$  we obtain the following:

$$U_m(x^{(m)}) = g_m^0(x_0) + \sum_{\substack{n=-m \\ n \neq 0}}^m \frac{g_m^{n,-n}(x_0)}{2!(N+1)} x_n x_{-n} + \sum_{\substack{n_1, \dots, n_4=-m \\ n_i \neq 0}}^m \frac{g_m^{n_1, \dots, n_4}(x_0)}{4!(N+1)^2} x_{n_1} x_{n_2} x_{n_3} x_{n_4} \delta_{n_1 + \dots + n_4, 0} + \dots \tag{23}$$

which contains many more coupling constants than equation (10): as usual each coupling constant is cutoff dependent but acquires now in addition a dependence on the Fourier modes

of the field. Then, most of these couplings disappear when the cutoff  $m$  reaches the value of its largest momentum. Note that in (23) there is still conservation of the momentum due to our choice of the potential (22).

The RG equation is still

$$S_{m-1}(x^{(m-1)}) = S_m(x^{(m-1)}) + \frac{1}{2} \log(\det(\mathcal{A})) - J^t \mathcal{A}^{-1} J \quad (24)$$

but the matrix  $\mathcal{A}$  is now

$$\mathcal{A} = \begin{pmatrix} 1 + \frac{U_m^{(m,-m)} + \text{Re}(U_m^{(m,m)})}{M\omega_m^2} & -\frac{\text{Im}(U_m^{(m,m)})}{M\omega_m^2} \\ -\frac{\text{Im}(U_m^{(m,m)})}{M\omega_m^2} & 1 + \frac{U_m^{(m,-m)} - \text{Re}(U_m^{(m,m)})}{M\omega_m^2} \end{pmatrix}$$

and

$$J = \begin{pmatrix} \text{Re}(U_m^{(m)}) \\ -\text{Im}(U_m^{(m)}) \end{pmatrix}$$

with the notation

$$U_m^{(n_1, \dots, n_p)} \equiv (N+1)^{\frac{p}{2}} \frac{\partial^p U_m}{\partial x_{n_1} \dots \partial x_{n_p}} \Big|_{(x_{n_1}, \dots, x_{n_p})=0}. \quad (25)$$

The flow equation for the potential is now

$$U_{m-1} = U_m + \frac{1}{2\beta} \lg \left( \left( 1 + \frac{U_m^{(m,-m)}}{M\omega_m^2} \right)^2 - \left| \frac{U_m^{(m,m)}}{M\omega_m^2} \right|^2 \right) - \frac{(M\omega_m^2 + U_m^{(m,-m)})|U_m^{(m)}|^2 - \text{Re}(U_m^{(-m,-m)}U_m^{(m)2})}{(M\omega_m^2 + U_m^{(m,-m)})^2 - |U_m^{(m,m)}|^2}. \quad (26)$$

The last contribution on the right-hand side originates from the ‘source term’. In this expression  $U_m$  and its various derivatives are functions of the path  $x^{(m-1)}$ . It is easy to check that the Taylor expansion of  $U_{m-1}$  is of the form (23). In addition, the right-hand side of (26) does not give any contributions to the kinetic term.

We deduce the equation for a constant-background path  $x_0$ , which now reads

$$U_{m-1}(x_0) = U_m(x_0) + \frac{1}{\beta} \log \left( 1 + \frac{U_m^{(m,-m)}(x_0)}{M\omega_m^2} \right). \quad (27)$$

Expanding (26) to the second order in  $x_0$  allows us to write for the second derivative of the potential

$$U_{m-1}^{(p,-p)}(x_0) = U_m^{(p,-p)}(x_0) + \frac{1}{\beta} \frac{U_m^{(p,-p,m,-m)}(x_0)}{M\omega_m^2 + U_m^{(m,-m)}(x_0)}. \quad (28)$$

In this formula the meaning of the derivative is similar to (25), that is

$$U_{m-1}^{(p,-p)}(x_0) = (N+1) \frac{\partial^2 U_m}{\partial x_p \partial x_{-p}} \Big|_{(x^{(m-1)})=x_0}. \quad (29)$$

Equations (27) and (28) are no longer inconsistent: one cannot obtain the second equation by deriving the first one because all the variables except  $x_0$  are already set to zero.

Note that the RG equation (27) for the potential is now different from (9) so new equations for the running coupling constants are expected.



### 5. Flow equations of the coupling constants

This time we choose to expand  $U$  around  $x_0 = 0$ . The  $n$ th coupling constant at the scale  $m$  is defined as

$$\delta_{n_1+\dots+n_p,0} g_m^{n_1,\dots,n_p} = (N+1)^{\frac{p}{2}} \left. \frac{\partial^p U_m}{\partial x_{n_1}, \dots, \partial x_{n_p}} \right|_0 \quad (30)$$

so that

$$U_m(x^{(m)}) = \sum_{p=0}^{\infty} \sum_{n_1,\dots,n_p=-m}^m \frac{g_m^{n_1,\dots,n_p}}{p!(N+1)^{\frac{p}{2}}} x_{n_1}, \dots, x_{n_p} \delta_{n_1+\dots+n_p,0}. \quad (31)$$

The equation for the first coupling constant is

$$g_{m-1}^0 = g_m^0 + \frac{1}{\beta} \log \left( 1 + \frac{g_m^{m,-m}}{M\omega_m^2} \right). \quad (32)$$

The value of the ground state energy is given by  $E_0 = g_0^0 = V_0(0)$ , which is the minimum of the effective potential. The flows of the quadratic coupling constants for  $p \leq m-1$  are

$$g_{m-1}^{p,-p} = g_m^{p,-p} + \frac{1}{\beta} \frac{g_m^{m,-m,p,-p}}{M\omega_m^2 + g_m^{m,-m}}. \quad (33)$$

The particular value  $g_0^{0,0}$  corresponds to the mass gap or the inverse correlation length in statistical mechanics language. Then it is well known that the first excited energy level can be deduced from the relation  $E_1 - E_0 = \sqrt{g_0^{0,0}}$ .

The flows of the fourth- and sixth-order couplings are given below:

$$g_{m-1}^{p_1,p_2,p_3,p_4} = g_m^{p_1,p_2,p_3,p_4} + \frac{1}{\beta} \left( \frac{g_m^{p_1,p_2,p_3,p_4,m,-m}}{M\omega_m^2 + g_m^{m,-m}} - \frac{g_m^{p_1,p_2,m,-m} g_m^{p_3,p_4,m,-m} + \text{Perm}(p_1, p_2, p_3, p_4)}{(M\omega_m^2 + g_m^{m,-m})^2} \right) \quad (34)$$

$$g_{m-1}^{p_1,p_2,p_3,p_4,p_5,p_6} = g_m^{p_1,p_2,p_3,p_4,p_5,p_6} + \frac{1}{\beta} \left( \frac{g_m^{p_1,p_2,p_3,p_4,p_5,p_6,m,-m}}{(M\omega_m^2 + g_m^{m,-m})} - \frac{g_m^{p_1,p_2,m,-m} g_m^{p_3,p_4,p_5,p_6,m,-m} + \text{Perm}}{(M\omega_m^2 + g_m^{m,-m})^2} + \frac{g_m^{p_1,p_2,m,-m} g_m^{p_3,p_4,m,-m} g_m^{p_5,p_6,m,-m} + \text{Perm}}{(M\omega_m^2 + g_m^{m,-m})^3} - \frac{g_m^{p_1,p_2,p_3,m} g_m^{p_4,p_5,p_6,-m} + \text{Perm}}{M\omega_m^2 + g_m^{m,-m}} \right) \quad (35)$$

where the last term is the tree level contribution of the ‘source term’. In these relations the conservation of the momenta is implicitly supposed as well as the condition that all the momenta are smaller than the cutoff. Apparently we see a different flow for each coupling constant. Some of them disappear when the cutoff reaches the largest momentum of the coupling. A tree level (source term) renormalization appears for the couplings bigger than the fourth order.

Suppose we want to compute the ground state energy of the anharmonic oscillator whose potential is

$$V_{\frac{N}{2}}(x) = \frac{M\Omega^2}{2} x^2 + \frac{\lambda}{4!} x^4. \quad (36)$$

For this initial potential, the coupling constants are Fourier mode independent. It is straightforward that the running coupling constants split into different families. For example  $g_m^{p,-p}$ ,  $g_m^{p,-p,q,-q}$ ,  $g_m^{p,-p,q,-q,r,-r}$  etc, are momentum independent. We can thus introduce the notation

$$g_m^{p,-p} = g_m^{(2)} \quad g_m^{p,-p,q,-q} = g_m^{(4)} \quad g_m^{p,-p,q,-q,r,-r} = g_m^{(6)} \quad \text{etc.} \quad (37)$$

To compute the ground state and the first excited state energy as done in [3], one can see that only the kind of coupling constant in (37) is necessary. This explains the great accuracy of the computation based on equation (9): the LPA gives precisely the flow of the coupling constants, allowing for the computation of the ground state energy. The other coupling constants follow different flows without influencing the preceding ones.

## 6. Renormalization of the kinetic term

In the preceding sections we have shown that the kinetic energy is RG invariant and that a mass term chosen to be position independent is constant along the flow. In this section we compute the flow equation of a generalized kinetic energy term. The corresponding action written in the continuum is now

$$S(x) = \int_0^{\hbar\beta} \left( \frac{1}{2} Z(x(t)) \left( \frac{d}{dt} x(t) \right)^2 + V(x(t)) \right). \quad (38)$$

At scale  $m$  we make the same ansatz for the potential energy as before. We write the kinetic term in the Fourier space as

$$\begin{aligned} & \frac{\epsilon}{\beta} \sum_{n=0}^{N+1} Z_m(x_0, e^{i\omega_1 t_n} x_1 + e^{-i\omega_1 t_n} x_{-1}, \dots, e^{i\omega_m t_n} x_m + e^{-i\omega_m t_n} x_{-m}) e^{i(\omega_i + \omega_j) t_n} \omega^i \omega^j x_i x_j \\ &= -Z_{m,i+j}(x^{(m)}) \omega^i x^i \omega^j x^j \end{aligned}$$

where the Fourier coefficient  $Z_{m,i+j}$  of  $Z_m(x^{(m)})$  is defined by

$$Z_{m,i+j} = \frac{\epsilon}{\beta} \sum_{n=0}^{N+1} Z_m e^{i(\omega_i + \omega_j) t_n}. \quad (39)$$

In order to follow as closely as possible the computation of the preceding section we introduce the notation for the coefficient  $\omega_m^2 x_m x_{-m}$ :

$$M_m \equiv Z_{m,0}(x^{(m-1)}). \quad (40)$$

The RG equation for the discretized action is then again

$$\exp\left(-\frac{1}{\hbar} S_{m-1}(x^{(m-1)})\right) = \int \frac{dx_m d\bar{x}_m}{\frac{2\pi\hbar}{\epsilon\omega_m M_m}} \exp\left(-\frac{1}{\hbar} S_m(x^{(m)})\right). \quad (41)$$

Again we compute the path integral by expanding the kinetic term to obtain all the terms quadratic in  $x_m$  and compute the Gaussian integral. The following conditions between the Fourier transform of the derivatives of the mass term are needed:

$$\begin{aligned} Z_{m,k}^{(m)} &= Z_{m,k}^{(-m)} \\ Z_{m,k}^{(m)} &\text{ is real} \\ Z_{m,k}^{(m,m)} &= \overline{Z_{m,k}^{(-m,-m)}} \\ Z_{m,-2m-i}^{(m)} &= -Z_{m,2m-i}^{(m)}. \end{aligned}$$

The RG equation is still given by (24) with the following matrices:

$$A = \begin{pmatrix} 1 + \frac{A_m^{(m,-m)} + \operatorname{Re}(B_m^{(m,m)})}{M_m \omega_m^2} & -\frac{\operatorname{Im}(B_m^{(m,m)})}{M_m \omega_m^2} \\ -\frac{\operatorname{Im}(B_m^{(m,m)})}{M_m \omega_m^2} & 1 + \frac{A_m^{(m,-m)} - \operatorname{Re}(B_m^{(m,m)})}{M_m \omega_m^2} \end{pmatrix} \quad (42)$$

$$J = \begin{pmatrix} \operatorname{Re}(C_m^{(m)}) \\ -\operatorname{Im}(C_m^{(m)}) \end{pmatrix} \quad (43)$$

where

$$A_m^{(m,-m)} = U_m^{(m,-m)} + Z_{m,i+j}^{(m,-m)} \omega^i x^i \omega^j x^j \quad (44)$$

and

$$B_m^{(m,m)} = U_m^{(m,m)} + Z_{m,2m} \omega_m^2 + Z_{m,2m+i+j}^{(m,m)} \omega^i x^i \omega^j x^j - \partial_j Z_{m,2m+i+j}^{(m)} \omega^i x^i \omega^j x^j - Z_{m,2m+i}^{(m)} (\omega^i)^2 x^i \quad (45)$$

and

$$C_m^{(m)} = U_m^{(m)} - Z_{m,m+i} (\omega^i)^2 x^i - \partial_j Z_{m,m+i+j} \omega^i x^i \omega^j x^j + Z_{m,m+i+j}^{(m)} \omega^i x^i \omega^j x^j \quad (46)$$

where all the terms  $A, B, C, U, Z$  are functions of the path  $x^{(m-1)}$ , and obviously the indices  $i, j$  are different from  $m$  or  $-m$ . The notation  $\partial_j Z$  means that  $x_j \neq 0$  as opposed to the notation  $Z^{(j)}$ .

With these notations the RG equation for the complete action reads

$$S_{m-1} = S_m + \frac{1}{2\beta} \lg \left( \left( 1 + \frac{A_m^{(m,-m)}}{M_m \omega_m^2} \right)^2 - \left| \frac{B_m^{(m,m)}}{M_m \omega_m^2} \right|^2 \right) - \frac{(M_m \omega_m^2 + A_m^{(m,-m)}) |C_m^{(m)}|^2 - \operatorname{Re}(B_m^{(m,m)} C_m^{(m)2})}{(M_m \omega_m^2 + A_m^{(m,-m)})^2 - |B_m^{(m,m)}|^2}. \quad (47)$$

We can now deduce the equation for the potential and the kinetic term. The contributions to the potential originate from terms which are  $\omega^i x^i \omega^j x^j$  independent. The flow equation of the mass term is found by keeping the contributions of the form  $\omega^i x^i \omega^j x^j$  and neglecting higher-order derivative interactions which are now generated by the RG. Remember that for a constant mass term there is no flow of the kinetic energy.

The RG equation for the kinetic term is

$$Z_{m-1,i+j} = Z_{m,i+j} + \frac{1}{2\beta} \left[ \frac{Z_{m,i+j}^{(m,-m)} (M_m \omega_m^2 + U_m^{(m,-m)})}{G} - \left( \frac{(Z_{m,i+j-2m}^{(-m,-m)} - \partial_j Z_{m,i+j-2m}^{(m)}) (U_m^{(m,m)} + Z_{m,2m} \omega_m^2)}{G} - \frac{(\partial_j Z_{m,2m} \omega_m^2 + \partial_j U_m^{(m,m)}) Z_{m,i-2m}^{(m)} + (Z_{m,2m} \omega_m^2 + U_m^{(m,m)}) \partial_j Z_{m,i-2m}^{(m)}}{G} + \operatorname{Perm}(m \leftrightarrow -m) \right) \right] + (\text{contribution of the source term}) \quad (48)$$

where

$$G = (M_m \omega_m^2 + U_m^{(m,-m)})^2 - |U_m^{(m,m)} + Z_{m,2m} \omega_m^2|^2. \quad (49)$$

The flow of the potential is now

$$U_{m-1} = U_m + \frac{1}{2\beta} \lg \left( \left( 1 + \frac{U_m^{(m,-m)}}{M_m \omega_m^2} \right)^2 - \left| \frac{U_m^{(m,m)}}{M_m \omega_m^2} \right|^2 \right) + (\text{contribution of the source term}). \quad (50)$$

The source term is quite complicated and not very illuminating; this is why we have not written it. Rather, by expanding the equations around  $x_0$ , we obtain the couple of equations concerning the potential and the kinetic term for a constant background field. In this case the source terms are zero. We obtain only the flow of the zero-component Fourier transform of  $Z$  denoted  $M_m \equiv Z_m(x_0)$ . This equation is deduced by considering (48) for  $i + j = 0$ ,

$$Z_{m-1}(x_0) = Z_m(x_0) + \frac{1}{2\beta} \left[ \frac{Z_m^{(m,-m)}(x_0)}{Z_m(x_0)\omega_m^2 + U_m^{(m,-m)}(x_0)} \right]. \tag{51}$$

Naturally we recover the equation for the effective potential:

$$U_{m-1}(x_0) = U_m(x_0) + \frac{1}{\beta} \log \left( 1 + \frac{U_m^{(m,-m)}(x_0)}{Z_m(x_0)\omega_m^2} \right). \tag{52}$$

Now recall that for large  $N$   $\omega_m^2$  is equivalent to  $(\frac{2\pi m}{\hbar\beta})^2$  and then runs from 0 to  $(\frac{\pi}{\epsilon})^2$ . Rewriting (52) as

$$U_{m-1}(x_0) = U_m(x_0) + \frac{\hbar}{2\pi} \frac{\pi}{\epsilon} \frac{2}{N+1} \log \left( 1 + \frac{U_m^{(m,-m)}(x_0)}{Z_m(x_0)(\frac{\pi}{\epsilon} \frac{2m}{N+1})^2} \right)$$

the limit  $N \rightarrow \infty$  (zero temperature) gives a continuous equation [3]

$$U_{k-\Delta k}(x_0) = U_k(x_0) + \frac{\hbar \Delta k}{2\pi} \log \left( 1 + \frac{U_k^{(k,-k)}(x_0)}{Z_k(x_0)k^2} \right) \tag{53}$$

where we have introduced the notations  $k^2 = \omega_m^2$ ,  $\Delta k = \frac{2\pi}{\epsilon(N+1)}$ . In the same manner we obtain a continuous equation for  $Z_k(x_0)$ , which is

$$Z_{k-\Delta k}(x_0) = Z_k(x_0) + \frac{\hbar \Delta k}{4\pi} \left[ \frac{Z_k^{(k,-k)}(x_0)}{Z_k(x_0)k^2 + U_k^{(k,-k)}(x_0)} \right]. \tag{54}$$

From (54) we see that if in the initial Lagrangian the mass term is constant ( $Z_{\frac{N}{2}, -\frac{N}{2}} = 0$ ) it will stay constant during the RG procedure as found in section 2, and no higher-derivative interactions will be generated (only the potential changes). This result is in contradiction with most of the results found in field theory. We discuss this point in the next section.

Note also that, by keeping a finite volume (finite temperature), we were able to integrate one mode after another. The continuous differential equations were derived by taking the infinite-volume limit only at the end of the computation. In the next section we derive again the RG equation by considering the infinite volume (zero temperature) from the beginning to show that non-analytical terms appear in this case.

### 7. Generalization to field theory

The point is the generation of non-analytic terms in field theory. These terms appear when the integration on the fast modes is performed on a spherical shell of thinness  $\Delta k$ . In fact doing again our computation in quantum mechanics directly in the continuum, we face exactly the same problem as in field theory. To simplify the computation, suppose that the background field is made up of a constant plus one mode configuration. Let us try to compute the RG equation for  $Z_k$  and  $U_k$ . For the point we want to show, we do not need to consider the generalized potential and it is enough to consider a position-independent mass term. The RG equation is defined as

$$\exp \left( -\frac{1}{\hbar} S_{k-\Delta k}(x) \right) = \int \mathcal{D}\tilde{x} \exp \left( -\frac{1}{\hbar} S_k(x + \tilde{x}) \right) \tag{55}$$

with the background path

$$x(t) = x_0 + \frac{1}{\beta} e^{iqt} x(q) + \text{h.c.} \tag{56}$$

and the fast fluctuating path

$$\tilde{x} = \int_k^{k-\Delta k} \frac{dp}{2\pi} e^{ipt} x(p) + \text{h.c.} \tag{57}$$

We arrive at the following equation, giving the flow of  $Z_k$  and  $U_k(x_0)$  [6]:

$$Z_{k-\Delta k} q^2 + U_{k-\Delta k}^{(2)}(x_0) = Z_k q^2 + U_k^{(2)}(x_0) + U_k^{(4)}(x_0) \int_k^{k-\Delta k} \frac{dp}{2\pi} \frac{1}{G(p)} + F(q) \tag{58}$$

with

$$F(q) = \frac{(U_k^{(3)})^2}{4} \int_k^{k-\Delta k} \frac{dp}{2\pi} \int_k^{k-\Delta k} \frac{dp'}{2\pi} \frac{\delta(p+p'+q) + \delta(p+p'-q)}{G(p)G(p')} + \text{h.c.} \tag{59}$$

It is clear that for  $q \leq \Delta k$  this integral gives a contribution of the form  $\Delta k - |q|$  because the domain of integration is deformed by the Dirac delta constraints. By expanding the denominator in powers of  $q$  we obtain

$$F(q) = \frac{(U_k^{(3)})^2}{2} \frac{\Delta k - |q|}{G(k)G(k)} (1 + O(q^2)) \quad \text{for } q \leq \Delta k \tag{60}$$

and

$$F(q) = 0 \quad \text{for } q \geq \Delta k. \tag{61}$$

We see the generation of non-analytic terms which plague the computation with a sharp cutoff, but, as we have seen in the preceding section, the integration mode after mode, which is an exact computation, gives  $F(q) = 0$ , so the generation of non-analytical terms is in fact only an artifact of the computation in the continuum and disappears if we perform a rigorous discrete calculus. The right result  $F(q) = 0$  can be recovered in the continuum if we suppose that  $q > \Delta k$ , because for that value of  $q$ ,  $F(q) = 0$  in the limit  $\Delta k \rightarrow 0$ . In this case, we face the problem already encountered of the inconsistency between the equation of the potential and its second derivative, but we have seen how to solve this problem in a very natural way with the generalized potential.

In dimension higher than one we always encounter trouble with the sharp cutoff formalism [4,9]. It would be natural to include a generalized potential, but terms such as  $F(q)$  are not necessarily equal to zero, because in field theory the domain of integration is connected whereas in quantum mechanics it is disconnected, and this may be the reason for the constant flow of  $Z_k$ . Equation (53) is the one-dimensional version of the Wegner–Houghton equation whereas (54) is very different from the equations obtained in field theory, for example in [6,8]. In these papers the non-analytical terms were simply neglected and a non-perturbative equation for  $Z_k$  was proposed by considering the case  $q \ll \Delta k$ . This way to treat the non-analytical terms has been already suggested in [11] and [12], but in our opinion none of the authors gave a convincing argument. The case  $q \ll \Delta k$  yields then (neglecting the non-analytical terms)

$$Z_{k-\Delta k} = Z_k + \frac{(U_k^{(3)})^2}{2} \frac{\Delta k}{G(k)G(k)}.$$

The motivation to consider this result as correct is that the equations for  $U_k(x_0)$  and  $U_k^{(2)}(x_0)$  are now consistent, but, firstly, it is mathematically meaningless to neglect the non-analytic terms [4]. Secondly, it is the merit of our computation to show without ambiguities that this reasoning is incorrect as the discrete calculus leads to  $Z_{k-\Delta k} = Z_k$ .

## 8. Conclusion

We have computed the RG equation for the running action in the context of a one-quantum-particle system. We have obtained this equation for an arbitrary background configuration by introducing a generalized action. This is necessary to obtain a closed form for the action under the RG flow. This action contains many more coupling constants than usual, and generates new flow equations for the coupling constants. An important point is that our construction allows us to avoid some inconsistencies arising in the continuous formulation. We plan to extend the generalized action construction in field theory.

## References

- [1] Wilson K G and Kogut I G 1974 *Phys. Rep.* **12** 75
- [2] Wegner F and Houghton H 1973 *Phys. Rev. A* **8** 401
- [3] Gosselin P, Grosdidier B and Mohrbach H 1999 *Phys. Lett. A* **256** 125
- [4] Morris T 1996 *Nucl. Phys. B* **458** 477
- [5] Kleinert H 1995 *Path Integrals in Quantum Mechanics Statistics and Polymer Physics* 2nd edn (Singapore: World Scientific)
- [6] Bonanno A *et al* 1999 *Phys. Rev. D* **60** 650091
- [7] Bonanno A and Zappalà D 1998 *Phys. Rev. D* **57** 7383
- [8] Branchina V 1999 *Preprint* hep-th/9912141
- [9] Wetterich C 1993 *Z. Phys. C* **57** 451
- [10] Ball R D *et al* 1995 *Phys. Lett. B* **347** 80
- [11] Wilson K G 1975 *Rev. Mod. Phys.* **47** 773
- [12] Sak J 1973 *Phys. Rev. B* **8** 281